

# Ay 201 Project Set 4 Solutions

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## Problem 1

### 1a

For LTE, we use the Saha equation for hydrogen:

$$\frac{n_p}{n_H} \approx 2 \left( \frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T}, \quad (1)$$

where  $n_H$  is the density of neutral hydrogen and  $n_p$  is the density of free protons. Using  $n_H = n - n_p$  and setting the right-hand side of the equation to  $\alpha(T)$ , we have

$$\frac{n_p^2}{n - n_p} \approx \alpha(T). \quad (2)$$

This can be solved using the quadratic equation by rearranging to

$$n_p^2 + \alpha n_p - \alpha n = 0. \quad (3)$$

Solving for  $n_p$  and dividing by  $n$  gives us the ionization fraction:

$$x_{\text{HII}} = -\frac{\alpha}{2n} + \frac{1}{2n} \sqrt{\alpha^2 + 4\alpha n} \quad (4)$$

(the quadratic also gives us a solution with a negative sign between the two terms, but we know  $x_{\text{HII}}$  must be a positive number, so we can ignore it). Plugging in numbers gives

$$\boxed{x_{\text{HII}} = 8.19 \times 10^{-5}.} \quad (5)$$

The optical depth is the density of electrons multiplied by the scale height and the electron-scattering cross-section:

$$\tau_{\text{es}} = x_{\text{HII}} n \sigma_{\text{T}} H = \boxed{5.44 \times 10^{-5}.} \quad (6)$$

## 1b

To get the free-free optical depth, use the free-free absorption coefficient and multiply by a characteristic scale height,  $H = 100$  km.

$$\tau_{\text{ff}} = \alpha_{\text{ff}} \Delta x \approx \alpha_{\text{ff}} H, \quad (7)$$

where we have  $\alpha$  from lecture:

$$\alpha_{\text{ff}} = 3.7 \times 10^8 Z^2 \frac{n_{\text{e}} n_i}{T^{1/2} \nu^3} (1 - e^{-h\nu/k_{\text{B}}T}) g_{\text{ff}}. \quad (8)$$

$n_i = n_e = x_{\text{HII}} n$ , we take  $g_{\text{ff}} \approx 1$ , and  $Z = 1$  for hydrogen.  $\nu = c/\lambda_0$ . This gives

$$\tau_{\text{ff}} = 3.7 \times 10^8 \frac{x_{\text{HII}}^2 n^2}{T^{1/2} \nu^3} (1 - e^{-h\nu/k_{\text{B}}T}) = \boxed{1.54 \times 10^{-5}.} \quad (9)$$

## 1c

We have the bound-free cross-section from the notes,

$$\sigma_{\text{bf}} \approx 6 \times 10^{-18} \frac{n}{Z^2} \left( \frac{\nu}{\nu_i} \right)^{-3}, \quad (10)$$

where  $\nu$  is still  $c/\lambda_0$  and  $\nu_i$  is the frequency required for ionization:

$$\nu_i = \frac{Z^2 Ry}{hn^2} \approx 3.29 \times 10^{15} \frac{Z^2}{n^2} \text{Hz}. \quad (11)$$

$Ry$  is one Rydberg = 13.6 eV. Note that in these two equations,  $n$  is the energy level of the atom. If  $\nu < \nu_i$ , the photons do not have enough energy to ionize the hydrogen, and the cross-section is zero.  $\nu = 6 \times 10^{14}$  Hz. For  $n = [1, 2, 3]$ ,  $\nu_i = [3.29, 0.82, 0.37] \times 10^{15}$  Hz, so  $\tau = 0$  for atoms in  $n = 1$  and  $n = 2$  levels, but for  $n = 3$  we have to use the formula. However, we first need to calculate the fraction of neutral atoms that will be in the  $n = 3$  state. We use the partition function and assume that all atoms are in the ground state of one of the first two excited states:

$$n_3 = n \left( \frac{g_3 e^{-E_3/kBT}}{g_1 e^{-E_1/kBT} + g_2 e^{-E_2/kBT} + g_3 e^{-E_3/kBT}} \right) \quad (12)$$

with  $E_i = Ry(1 - 1/n_i^2)$  and  $g_i = 2n_i^2$ . The density we get is  $n_3 = 8.27 \times 10^{-12}n$ , so we plug this in for

$$\tau_{bf} = n_3 \sigma_{bf} H = \boxed{3.03 \times 10^{-4}}. \quad (13)$$

## 1d

We can treat this case as similar to normal hydrogen ionization, except this time  $\chi = 0.75$  eV =  $1.2 \times 10^{-12}$  ergs. Let's assume basically all hydrogen atoms are either neutral or are  $H^-$  ions (since the ionization fraction we found was very small) and use Saha:

$$\frac{n_H}{n_{H^-}} = \frac{g_0}{g_-} \frac{2}{n_e} \left( \frac{2\pi m_e k_B T}{h^2} \right)^{3/2} e^{-\chi/k_B T} \quad (14)$$

where the ratio of the degeneracies is about 1. Substituting  $n_e = x_{HII} n_H$ , we get

$$n_{H^-} = \frac{n_H^2 x_{HII}}{2} \left( \frac{2\pi m_e k_B T}{h^2} \right)^{-3/2} e^{\chi/k_B T} = 4.05 \times 10^9 \text{ cm}^{-3}. \quad (15)$$

Now assume the neutral hydrogen is in the ground state (as we saw, most of it is). Again use  $\sigma_{bf} \approx 6 \times 10^{-18} \frac{n}{Z^2} \left( \frac{\nu}{\nu_i} \right)^{-3}$ .  $\nu_i = 1.8 \times 10^{14}$  Hz this time, and  $\nu = 6 \times 10^{14}$  Hz, so the cross section is nonzero. Plugging in numbers, we get

$$\sigma_{\text{bf}} = 1.66 \times 10^{-19} \text{cm}^2 \longrightarrow \tau = \sigma_{\text{bf}} n_{\text{H}^-} H = 0.0067. \quad (16)$$

This optical depth is small, but it still clearly dominates. This opacity is about two orders of magnitude or more than each of the other sources of opacity.

## Problem 2

### 2a

We have the cooling time for hydrogen from class:

$$t_c = 9 \times 10^{10} T^{1/2} n_{\text{I}}^{-1} \text{sec}. \quad (17)$$

The problem says the mass of hydrogen gas is on the order of the dark matter mass, so let's assume  $M$  is the mass of hydrogen. This gives us  $n_{\text{I}} = \frac{M}{m_{\text{p}}} \frac{3}{4\pi R^3}$ . Setting  $t_c = t_{\text{dyn}}$  gives the expression

$$\left[ \frac{GM}{R^3} \right]^{-1/2} \approx 9 \times 10^{10} T_{\text{v}}^{1/2} \left[ \frac{4\pi m_{\text{p}} R^3}{3M} \right]. \quad (18)$$

Doing algebra and using the definition of the virial temperature to get  $M \sim k_{\text{B}} T_{\text{v}} R / G m_{\text{p}}$ , we find that the temperature cancels and we get the expression for  $R_{\text{g}}$ :

$$R_{\text{g}} \approx \frac{3k_{\text{B}}}{4\pi m_{\text{p}}^{3/2} 9 \times 10^{10}} = 2.16 \times 10^{23} \text{cm} = 70.2 \text{kpc}, \quad (19)$$

so we indeed find that the radius is on the order of 80 kpc.

### 2b

Free-free emission is dependent on the amount of free electrons and protons present in the gas. Therefore, we need to calculate the fraction of ions given a certain temperature and the assumption of CIE (collisional ionization equilibrium):

$$n_{\text{HI}} C_{ic} = n_{\text{HII}} \alpha_A n_{\text{e}} \quad (20)$$

From this we can derive that

$$y_{\text{II}} = \frac{n_{\text{II}}}{n_{\text{I}}} = \frac{C_{ic}}{\alpha_A n_e} = 1.3 \times 10^{15} \left( \frac{k}{\chi} \right)^2 \left( \frac{T}{T_4} \right) e^{-\chi/kT} \quad (21)$$

And we can write

$$x_{\text{II}} = \frac{n_{\text{II}}}{n_{\text{H}}} = \frac{n_{\text{II}}}{n_{\text{I}} + n_{\text{II}}} = \frac{y_{\text{II}}}{1 + y_{\text{II}}} \quad (22)$$

These will be important when writing out the electron and proton densities in what follows.

The emission is given by

$$\epsilon_{\text{ff}} = 1.4 \times 10^{-27} Z^2 n_e n_{\text{II}} T^{1/2} g_{\text{ff}} \quad (23)$$

We have already calculated  $n_{\text{II}}$  above. Assume that  $g_{\text{ff}} \approx 1$ , set  $Z = 1$  for Hydrogen, and convert to dimensionless units. Then

$$\boxed{\Lambda_{\text{ff}} = \frac{\epsilon}{n_e n_{\text{H}}} = 1.4 \times 10^{-25} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left( \frac{T}{T_4} \right)^{1/2} \frac{\text{erg cm}^3}{\text{s}}} \quad (24)$$

For bound-free emission, again the proton and electron fractions are important. The emission is

$$\epsilon_{\text{bf}} = 3.25 \times 10^{-13} n_e n_{\text{I}} k_{\text{B}} T \left( \frac{T}{T_4} \right)^{-1/2} \quad (25)$$

Convert to dimensionless units. Then

$$\boxed{\Lambda_{\text{bf}} = \frac{\epsilon}{n_e n_{\text{H}}} = 5 \times 10^{-25} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left( \frac{T}{T_4} \right)^{1/2} \frac{\text{erg cm}^3}{\text{s}}} \quad (26)$$

For Lyman- $\alpha$  emission,

$$\epsilon_{\text{Ly}-\alpha} \approx h\nu_0 n_2 A_{21} = h\nu_0 n_1 C_{12} \quad (27)$$

where  $\nu_0$  is the frequency associated with the Lyman- $\alpha$  transition,  $A_{21}$  is the rate of transitions from  $n = 2 \rightarrow n = 1$ , and  $C_{12}$  is the rate of collisional excitation from  $n = 1 \rightarrow n = 2$ .

Using the formulae derived in class, we can rewrite  $C_{12}$  in the final expression to get

$$\epsilon_{\text{Ly}-\alpha} \approx 2.16 h \nu_0 n_1 n_e f \left( \frac{h \nu_0}{kT} \right)^{-1.68} T^{-3/2} e^{-h \nu_0 / kT} \quad (28)$$

where  $f \approx .5$  is the oscillator strength of the Lyman- $\alpha$  transition and  $n_1$  is the fraction of neutral Hydrogen atoms in the ground state (assume  $n_1 \approx n_{\text{I}}$ ). Converting into dimensionless variables, we have

$$\Lambda_{\text{Ly}-\alpha} = \frac{\epsilon}{n_e n_{\text{H}}} = 4 \times 10^{-18} \frac{y_{\text{II}}}{1 + y_{\text{II}}} \left( \frac{h \nu_0}{kT} \right)^{-.68} \left( \frac{T}{T_4} \right)^{-1/2} \frac{\text{erg cm}^3}{\text{s}} \quad (29)$$

To calculate the fraction of neutral Hydrogen, write

$$\frac{n_{\text{I}}}{n_{\text{tot}}} = \frac{n_{\text{I}}}{n_{\text{I}} + n_{\text{II}}} = \frac{1}{1 + y_{n_{\text{II}}}} \quad (30)$$

So putting these together,

$$\boxed{\Lambda_{\text{Ly}-\alpha} = \frac{\epsilon}{n_e n_{\text{H}}} = 4 \times 10^{-18} \frac{1}{1 + y_{n_{\text{II}}}} \left( \frac{h \nu_0}{kT} \right)^{-.68} \left( \frac{T}{T_4} \right)^{-1/2} \frac{\text{erg cm}^3}{\text{s}}} \quad (31)$$

## 2c

The probability of absorption into the thermal pool is just the probability of collisional de-excitation. We can write this as

$$P = \frac{C_{21}}{C_{21} + A_{21}} \quad (32)$$

where  $A_{21} = 6.3 \times 10^8 \text{s}^{-1}$  (from NIST) and at  $T = T_4$ ,  $n_e = 1 \text{cm}^{-3}$ ,

$$C_{21} = 5 \times 10^{-3} \text{s}^{-1} \quad (33)$$

Plugging these values into the above equation, we have

$$P = \frac{C_{21}}{C_{21} + A_{21}} \approx \boxed{10^{-11}} \quad (34)$$

So there is a very small probability that absorption into the thermal pool will take place.

## 2d

We solve the following balance equations

$$n_5 C_{56} = n_6 R_{65} \quad (35)$$

$$n_6 C_{67} = n_7 R_{76} \quad (36)$$

$$n_{\text{oxy}} = n_5 + n_6 + n_7 \quad (37)$$

First divide the latter equation by  $n_6$  and invert to get

$$\frac{n_6}{n_{\text{oxy}}} = \left[ \frac{n_5}{n_6} + 1 + \frac{n_7}{n_6} \right]^{-1} \quad (38)$$

Now plug in the relations from above to arrive at  $\boxed{\frac{n_6}{n_{\text{oxy}}} = \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1}}$

## 2e

Look at the Grotrian diagram for the OVI ion. You can see that the first excited state is a short energy step above ground state ( $\Delta E = 12\text{eV}$ ). From this we can confidently say that mostly the ground and first excited states will be populated. The cooling will therefore be dominated by the transitions  $n = 2 \rightarrow n = 1$ .

For OVII, the lowest energy level transition requires a  $\lambda = 22\text{\AA}$  photon to excite the electron to that level,, or a gas temperature of  $\Delta E/k = 6 \times 10^6$  Kelvins. This is outside of our temperature range, so we can neglect this temperature.

For OVIII, the lowest energy level transition requires a  $\lambda = 18\text{\AA}$  photon to excite the electron to that level,, or a gas temperature of  $\Delta E/k = 8 \times 10^6$  Kelvins. This is outside of our temperature range, so we can neglect this temperature.

But a transition for OV only requires a  $\lambda = 1218\text{\AA}$  photon to excite the electron to that level, or a gas temperature of  $\Delta E/k = 1 \times 10^5$  Kelvins. Plenty of particles will have enough energy to impart collisionally to excite this transition.

## 2f

Here we derive the cooling function for the line transition of OVII. We assume CIE again. Start with the expression for emission:

$$\epsilon_{\text{OVI}} = n_2 A_{21} h \nu_0 \quad (39)$$

where  $\nu_0$  is the frequency associated with the line transition and  $n_2$  is the number density of particles in the excited state of OVI. Assuming collisional de-excitation is a negligible process. Then

$$n_2 A_{21} = n_1 C_{12} = (n_{\text{OVI}} - n_2) C_{12} \quad (40)$$

Rearranging:

$$n_2 = n_{\text{OVI}} \frac{C_{12}}{C_{12} + A_{21}} \quad (41)$$

And recall from section (2d)

$$\frac{n_{\text{OVI}}}{n_{\text{oxy}}} = \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \quad (42)$$

Assuming solar abundances, we can make the following conversion

$$\frac{n_{\text{OVI}}}{n_{\text{oxy}}} = \frac{X_{\text{H,sol}}}{X_{\text{oxy,sol}}} \frac{n_{\text{OVI}}}{n_{\text{H}}} \quad (43)$$

where we have defined  $X_{\text{oxy,sol}} = n_{\text{oxy}}/n_{\text{tot}}$  in the sun. Thus, finally, we can write

$$n_2 = n_{\text{H}} \frac{X_{\text{oxy,sol}}}{X_{\text{H,sol}}} \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \frac{C_{12}}{C_{12} + A_{21}} \quad (44)$$

Putting it all together,

$$\Lambda_{\text{OVI}} = \frac{\epsilon}{n_e n_{\text{H}}} = h \nu_0 A_{21} \frac{X_{\text{oxy,sol}}}{X_{\text{H,sol}}} \left[ \frac{R_{65}}{C_{56}} + 1 + \frac{C_{67}}{R_{76}} \right]^{-1} \frac{C_{12}}{C_{12} + A_{21}} \frac{\text{erg cm}^3}{\text{s}} \quad (45)$$

where

$$C_{12} \approx 3.9 n_e f \left[ \frac{h \nu_0}{kT} \right]^{-1} T^{-3/2} e^{-h \nu_0 / kT}$$



for  $\nu_0$  corresponding to the line transition,

$$C_{56} = 2.7n_e \left[ \frac{\chi_{56}}{kT} \right] T^{-3/2} e^{-\chi_{56}/kT}$$

$$C_{67} = 2.7n_e \left[ \frac{\chi_{67}}{kT} \right] T^{-3/2} e^{-\chi_{67}/kT}$$

$$R_{65} = 2 \times 10^{-13} Z^2 (T/T_4)^{-1/2}$$

for  $Z=7$ ,

$$R_{76} = 2 \times 10^{-13} Z^2 (T/T_4)^{-1/2}$$

for  $Z=6$ .

The cooling functions from section (2b) and (2f) are plotted in Fig. 1.

Figure 1: Cooling function. Courtesy of J.L. Barnes.

